# Acoustic identification of a poroelastic cylinder

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**Résumé**

We show how to cope with the acoustic identification of poroelastic materials when the specimen is in the form of a cylinder. We apply our formulation, based on the Biot model, approximated by the equivalent elastic solid model, to a long bone-like or borehole sample specimen probed by low frequency sound.

## 1 Introduction

It has become fairly common [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], to acoustically identify the structural/material properties, and/or the phase velocity/attenuation in poroelastic (e.g., biological or geophysical) materials by processing data relative to the reflected and/or transmitted pulses of a slab-like specimen of the material in response to an incident plane-wave pulse. In most of these studies, the model of the medium is that of a fluid or (less often) viscoelastic solid. It is not often that materials, especially those of biological [16] or geophysical [17] nature, present themselves in the form of slabs, plates or layers, nor is it judicious to cut them (an operation that is rarely accurate and which often modifies the physical properties) to fit this shape. We show herein how to cope with the acoustic identification of these materials, in the form of cylindrical specimens [16], [17], whether they are considered to be fluid-like or elastic solid-like. Work on the fully-poroelastic (in the sense of Biot) case is in progress.

More specifically, this investigation is concerned with the reconstruction of the material constants $\lambda^1 = 1/\kappa^1$ ($\kappa^1$ the complex compressibility), $\mu^1$ (shear modulus), $\rho^1$ (density) of an elastic solid-like (assumed to be of this nature even if the target is poroelastic in the sense of Biot), almost-circular cylinder, modeled as a circular cylinder of radius $a$ close to the average radius of the almost-circular cylindrical target. The latter is immersed in a fluid-like host wherein propagates plane-wave like acoustic probe radiation. The action of this wave on the target results in a scattered acoustic wavefield which serves to reveal the material properties of the target.

The wavelength ($\Lambda_L^0 = 2\pi/k_L^0$) of the probe radiation is assumed to be much larger ($\chi_L^0 := k_L^0 a << 1$) than $a$. A perturbation analysis is shown to enable an explicit reconstruction of $\rho^1$ and of a linear combination of $\lambda^1$, $\mu^1$.

This reconstruction technique relies on the a priori knowledge of $a$ ; it is assumed herein that this parameter is known.

## 2 Physical configuration

The scattering body is an infinite cylinder whose generators are parallel to the $z$ axis in the cylindrical coordinate system $(r, \theta, z)$. The intersection of the cylinder, within which is located the origin $O$, with the $xOy$ plane defines :

i) the boundary curve $\Gamma = \{r = f(\theta); 0 \leq \theta < 2\pi\}$, with $f$ a continuous, single-valued function of $\theta$ ; further on, we shall take $\Gamma$ to be a circle, i.e., $f(\theta) = a$, with $a$ its radius, close to the average value of $\eta(\theta)$,

ii) the bounded (inner) region (i.e., the one occupied by the body in its cross-section plane) $\Omega_1 = \{r < \eta(\theta);\ 0 \leq \theta < 2\pi\}$,

iii) the unbounded (outer) region $\Omega_0 = \{r > \eta(\theta);\ 0 \leq \theta < 2\pi\}$.

It is assumed that $\Omega_0$ is filled with a linear, homogeneous, inviscid fluid $M^0$ and $\Omega_1$ by a linear, macroscopically-homogeneous, isotropic, porous medium $M^1$ which will subsequently be associated with a linear, homogeneous, isotropic, time-invariant elastic solid medium.

The material constants of $M^0$ are assumed to be known. Those of $M^1$ are unknown and are to be recovered by the technique described hereafter. The latter relies on probing the cylinder (from the outside) by a plane acoustic wave whose wavevector lies in the $xOy$ plane.

# 3 Ingredients of the Biot theory

We give the ingredients of the basic Biot theory of biphasic (solid/fluid) porous media [18], [19].

## 3.1 Conservation of momentum relations

In the absence of applied body forces, the conservation of momentum relations take the form

$$\nabla \cdot \boldsymbol{\sigma} - \rho \mathbf{u} - \rho_f \mathbf{w}_{,tt} = \mathbf{0} \quad , \quad \nabla p + \rho_f \mathbf{u}_{,tt} + m \mathbf{w}_{,tt} + \frac{\eta F}{\kappa} \mathbf{w}_{,t} = 0 \, , \tag{1}$$

wherein :

- $\mathbf{f}_{,t}$ designates a first-order partial derivative with respect to time $t$, and $\mathbf{f}_{,tt} := (f_{,t})_{,t}$,
- $\boldsymbol{\sigma}$ is the total stress tensor in the porous fluid-saturated medium,
- $\rho_s$ the density of the solid component
- $\rho_f$ the density of the viscous fluid filling the (interconnected) pores,
- $\rho$ the bulk density of the porous medium, such that $\rho = (1-\phi)\rho_s + \phi\rho_f$, with $\phi$ the porosity (volume fraction of fluid relative to total volume in a representative volume element),
- $\boldsymbol{\sigma}$ the stress tensor,
- $\mathbf{u}$ the displacement vector of the solid particle component,
- $\mathbf{U}$ the fluid particle displacement vector,
- $\mathbf{w}$ the relative displacement vector of the fluid particle relative to the solid particle (both particles assumed to occupy the same point) defined as $\mathbf{w} := \phi(\mathbf{U} - \mathbf{u})$,
- $p$ the pressure in the fluid component of the porous medium,
- $\eta$ the viscosity of this fluid,
- $\kappa$ the (low-frequency) permeability,
- $m = \frac{\rho_f \alpha}{\phi}$ the virtual mass,
- $\alpha$ the tortuosity (in [20] $\alpha$ is termed the virtual mass coefficient or structure factor), which, in [21], is related to the porosity by $\alpha = 1 + r\left(\frac{1-\phi}{\phi}\right)$ (wherein $r$ is a constant with a value between 0 and 1),
- $F(t)$ is a linear integral convolution operator with respect to time which, in the frequency domain, becomes a frequency-dependent multiplier $F(\omega)$, implying frequency-dependent permeability, i.e., $\tilde{\kappa}(\omega) = \frac{\kappa}{F(\omega)}$, wherein $\tilde{\kappa}(\omega)$ is the so-called dynamic permeability, and $F$ is designed so that $\lim_{\omega \to 0} F(\omega) = 1$.

## 3.2 Constitutive relations

Biot's constitutive relations [18],[22] linearly relate the total stress and fluid pressure to the (isotropic) solid and fluid displacement spatial derivatives via

$$\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon} + \left[(H - 2\mu)e - C\zeta\right]\mathbf{I} \quad , \quad p = M\zeta - Ce \tag{2}$$

with :

$$D = K_s\left[1 + \phi\left(\frac{K_s}{K_f} - 1\right)\right] \quad , \quad H = \frac{(K_s - K)^2}{D - K} + K + \frac{4\mu}{3} \, , \tag{3}$$

$$C = \frac{K_s(K_s - K)}{D - K} \quad , \quad M = \frac{K_s^2}{D - K} \quad , \quad e = \nabla \cdot \mathbf{u} \ , \ \zeta = -\nabla \cdot \mathbf{w} \, , \tag{4}$$

wherein :

- $\boldsymbol{\varepsilon}$ is the strain tensor $\boldsymbol{\varepsilon} = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right)$,
- $\mathbf{I}$ the unit tensor
- $\mu$ the shear modulus (rigidity) of the saturated solid,
- $K = (\lambda + \frac{2}{3}\mu)$, $K_s$ the bulk moduli of the dry (i.e., drained) solid matrix and solid grain material respectively (note that $\lambda_c = \left(1 - \frac{K}{K_s}\right)C + \lambda$ is the Lamé constant of the saturated solid),
- $K_f$ the fluid bulk modulus.

### 3.3 Equations of motion in terms of $\mathbf{u}$ and $\mathbf{w}$

We assume henceforth that all the material parameters are constants with respect to position, i.e., *the medium is macroscopically homogeneous*. The conservation of momentum equations and constitutive relations are employed in such a way as to eliminate the pressure and stress tensor so as to obtain :

$$\mu\nabla^2\mathbf{u} + (H-\mu)\nabla e - C\nabla\zeta = \rho\mathbf{u}_{tt} + \rho_f\mathbf{w}_{,tt} \quad , \quad C\nabla e - M\nabla\zeta = \rho_f\mathbf{u}_{,tt} + m\mathbf{w}_{,tt} + \frac{\eta}{\kappa}F\mathbf{w}_{,t} \ , \tag{5}$$

which is the vectorial form of the Biot wave equations as given by Yamamoto [19] for a macroscopically-homogeneous porous medium.

# 4 Choice of an approximate model to describe wave propagation in a porous medium

The fundamental difficulty with the Biot theory is twofold : i) *two* coupled (vectorial) wave equations have to be solved simultaneously, and ii) many material parameters have to be recovered in the inverse problem context. This is why the traditional approach (notably in the underwater acoustics community) has been to reduce this model to a simpler one (with fewer material parameters) in which only one (vectorial or scalar) wave equation has to be solved.

### 4.1 Equivalent elastic solid model (EESM) obtained from the limit $\phi \to 0$

The introduction of the second equation of (5) into the first gives

$$\mu\nabla^2\mathbf{u} + \left(H-\mu-\frac{C}{m}\right)\nabla e + \left(-C+\frac{M}{m}\right)\nabla\zeta + \left(-\rho+\frac{\rho_f}{m}\right)\mathbf{u}_{tt} + \frac{\eta}{m\kappa}F\mathbf{w}_{,t} = \mathbf{0} \ , \tag{6}$$

Recalling the definitions of $m$ and $\mathbf{w}$, we conclude that $\lim_{\phi\to 0} \ m^{-1} = 0$ and $\lim_{\phi\to 0} \ \mathbf{w} = \mathbf{0}$, so that in the limit $\phi \to 0$, (6) becomes

$$\mu\nabla^2\mathbf{u} + (\lambda_c+\mu)\,\nabla\nabla\cdot\mathbf{u} - \rho\mathbf{u}_{tt} = \mathbf{0} \ , \tag{7}$$

which is simply the Navier wave equation in a non-dissipative, linear, homogeneous, isotropic solid in which the material parameters are $\lambda_c, \ \mu, \ \rho = \rho_s$.

This equation forms the basis of the so-called equivalent elastic solid model for wave propagation in poroelastic media, often employed for the evaluation of transmission loss of sound over sediment layers on sea bottoms [23], [24].

### 4.2 Equivalent viscoelastic solid model (EVSM) for wave propagation in poroelastic media

We now consider another approximation of the Biot wave equations which leads to what has been termed the equivalent viscoelastic solid model of propagation in poroelastic media [25][22].

The basic idea is to reduce the Biot model to only one wave equation, while retaining the loss mechanism inherent in this model (contrary to what is done in the equivalent elastic solid model in which the loss mechanism is abolished in the limit $\phi \to 0$).

Let us return to the definition of $\mathbf{w}$, i.e., $\mathbf{w} = \phi(\mathbf{U}-\mathbf{u})$ in which we assume $\phi\mathbf{U} << \phi\mathbf{u}$. Thus, neglecting the terms in $\phi\mathbf{U}$ in (6), we get

$$\mu\nabla^2\mathbf{u} + \left[H-\mu-\frac{C}{m}-C\phi+\frac{M\phi}{m}\right]\nabla\nabla\cdot\mathbf{u} + \left(-\rho+\frac{\rho_f}{m}\right)\mathbf{u}_{tt} - \frac{\eta\phi}{m\kappa}F\mathbf{u}_{,t} = \mathbf{0} \ . \tag{8}$$

This wave equation for the displacement in the solid component is similar to the Navier equation, with the exception that the wavenumber is now complex, its real part not being equal to that of the Navier equation wavenumber, and its imaginary part being conditioned by $\frac{\eta\phi}{m\kappa}F$.

### 4.3 Equivalent fluid model (EFM)

The equivalent fluid model (EFM) is appropriate when the fluid is light (e.g., a gas such as air) and the solid skeleton is therefore relatively immobile (i.e., rigid). This model has been employed, even when the fluid is not light, notably in the underwater acoustics community [26], [23], [19].

There exist various versions of the EFM, several of which are described and compared in Depollier *et al.* [27]. The one we shall consider herein is a simplified version of the model offered in [11].

No restrictions are introduced concerning the porosity, but, for the sake of simplicity, the fluid viscosity is assumed to be nil. The fundamental assumption is that the solid component is rigid, i.e.

$$K_s = \infty, \quad , \quad \mathbf{u} = 0 \ \ \rightarrow \mathbf{u}_{,tt} = 0 \, . \tag{9}$$

If, in addition, the medium is macroscopically-homogeneous and time-invariant (i.e., $\rho_f$, $\alpha$ and $K_f$ are constants with respect to position and $t$) then the Biot system of equations reduce to

$$-\nabla^2 p - \rho_f \alpha \nabla \cdot \mathbf{U}_{,tt} = 0 \quad , \quad p_{,tt} + K_f \mathbf{U}_{,tt} = 0 \, , \tag{10}$$

a linear combination of which yields

$$\nabla^2 p - \frac{\alpha \rho_f}{K_f} p_{,tt} = 0 \, , \tag{11}$$

which is the wave equation in the equivalent fluid.

Note that $\alpha = 1$ for a homogeneous fluid.

### 4.4 Our choice of approximate model

Since all three approximate models reduce to the same type of wave equation (either vectorial or scalar), a generic choice–(vectorial in nature) adopted herein–is the EESM.

This means that we replace, by thought, the porous medium cylinder immersed in an inviscid fluid by an elastic solid cylinder immersed in the same fluid.

## 5 Mathematical description of the problem

### 5.1 Preliminaries

Due to the invariance of the cylinder and incident wavefield $p^i$ with respect to $z$, the incident and scattered fields are also invariant with respect to $z$.

Let $p^0$ designate pressure in $\Omega_0$ ; due to this invariance, $p^0 = p(x,y,t) = p^i(x,y,t) + p^d(x,y,t)$, wherein $p^d$ is the diffracted pressure in $\Omega_0$.

For the same reason, the total displacement wavefield $\mathbf{u}$ in $\Omega_1$ is of the form $\mathbf{u} = \mathbf{u}(x,y,t)$.

The analysis is carried out in the space-frequency domain via $p^{0,i,d}(\mathbf{x},t) = \int_{-\infty}^{\infty} p^{0,i,d}(\mathbf{x},\omega) \exp(-i\omega t) d\omega$ and $\mathbf{u}^1(\mathbf{x},t) = \int_{-\infty}^{\infty} \mathbf{u}^1(\mathbf{x},\omega) \exp(-i\omega t) d\omega$ wherein $\mathbf{x} = (x,y)$.

Henceforth, it is implicit that $p^{0,i,d}$ (meaning $p^0$, $p^i$ or $p^d$) and $\mathbf{u}^1$ are functions of $(\mathbf{x},\omega)$.

### 5.2 Governing equations

The incident plane wave is

$$p^i = S(\omega) \exp\left(-ik_0 r \cos\left(\theta - \theta^i\right)\right) \tag{12}$$

(wherein $S(\omega)$ is the amplitude spectrum, $\theta^i$ the incident angle in the $xOy$ plane, and $k^0 = \omega / c^0$, with $c^0 = (\rho^0 \kappa^0)^{-1/2}$).

$p^{0,i,d}$ satisfy the frequency-domain pressure wave (Helmholtz) equation

$$\left(\nabla^2 + (k_L^0)^2\right) p^{0,i,d} = 0 \quad \text{in } \Omega_0 \;, \tag{13}$$

and the radiation condition

$$p^d_{,r} - ik^0 p^d = o(r^{-1/2}) \; ; \; r \to \infty \; , \; \forall\theta \in [0, 2\pi[ \; . \tag{14}$$

$\mathbf{u}^1$ satisfies the frequency-domain elastic wave equation

$$\mu\nabla^2\mathbf{u}^1 + (\lambda^1 + \mu^1)\nabla\cdot\nabla\mathbf{u}^1 + \omega^2\mathbf{u}^1 = 0 \quad \text{in } \Omega_1 \;, \tag{15}$$

and the boundedness condition

$$\|u^1\| < \infty \qquad \text{in } \Omega_1 \; . \tag{16}$$

Let $\boldsymbol{\nu}$ designate the unit outward-pointing (from $\Omega^1$) unit normal vector, $\mathbf{T}^j = \boldsymbol{\sigma}^j \cdot \boldsymbol{\nu}$ the traction. Then the transmission boundary conditions are :

$$\mathbf{T}^0 - \mathbf{T}^1 = \mathbf{0} \quad , \quad \mathbf{u}^0\cdot\boldsymbol{\nu} - \mathbf{u}^1\cdot\boldsymbol{\nu} = 0 \qquad \text{on } \Gamma \; . \tag{17}$$

## 5.3 Reduction of the elastic solid wave equation to two Helmholtz equations

The use of the Helmholtz decomposition

$$\mathbf{u}^1 = \nabla\varphi^1 + \nabla\times\boldsymbol{\psi}^1 \;, \tag{18}$$

enables (15) to be reduced to the two (one scalar, the other vectorial) Helmholtz equations

$$\left(\nabla^2 + (k_L^1)^2\right)\varphi^1 = 0 \quad , \quad \left(\nabla^2 + (k_T^1)^2\right)\boldsymbol{\psi}^1 = \mathbf{0} \;, \tag{19}$$

wherein

$$k_L^1 = \frac{\omega}{c_L^1} = \omega\left(\frac{\lambda^1 + 2\mu^1}{\rho^1}\right)^{-1/2} \quad , \quad k_T^1 = \frac{\omega}{c_T^1} = \omega\left(\frac{\mu^1}{\rho^1}\right)^{-1/2} \; . \tag{20}$$

Recalling that the fields $p$ and $\mathbf{u}$ do not depend on $z$ enables (13) and (19) to be cast into the cylindrical coordinate forms :

$$\begin{gathered} p^0_{,rr} + r^{-1}p^0_{,r} + r^{-2}p^0_{,\theta\theta} + (k_L^0)^2p^0 = 0 \quad , \quad \varphi^1_{,rr} + r^{-1}\varphi^1_{,r} + r^{-2}\varphi^1_{,\theta\theta} + (k_L^1)^2\varphi^1 = 0 \\ \boldsymbol{\psi}^1_{,rr} + r^{-1}\boldsymbol{\psi}^1_{,r} + r^{-2}\boldsymbol{\psi}^1_{,\theta\theta} + (k_T^1)^2\boldsymbol{\psi}^1 = 0 \; . \end{gathered} \tag{21}$$

The gauge condition $\nabla\cdot\boldsymbol{\psi}^1 = 0$ and the absence of shear stress in the fluid imply that $\psi_r^1 = \psi_\theta^1 = 0$. In the cylindrical coordinate system, the traction and normal component of displacement continuity conditions reduce to :

$$\begin{gathered} -p^0 + \lambda^1(k_L^1)^2\varphi^1 - 2\mu^1\left[\varphi^1_{,rr} - r^{-2}\psi^1_{z,\theta} + r^{-1}\psi^1_{z,r\theta}\right] = 0 \;, \\ 2\left[-r^{-2}\varphi^1_{,\theta} + r^{-1}\varphi^1_{,r\theta}\right] + \left[-\psi^1_{z,rr} + r^{-1}\psi^1_{z,r} + r^{-2}\psi^1_{z,\theta\theta}\right] = 0 \;, \\ \frac{1}{\lambda^0(k_L^0)^2}p^0_{,r} - \phi^1_{,r} - r^{-1}\psi^1_{z,\theta} = 0 \; . \end{gathered} \tag{22}$$

### 5.4 Field representations

The incident pressure field satisfies the periodicity condition $p^i(r,-\theta+2\theta^i,\omega)=p^i(r,\theta,\omega)$ and the first of the Helmholtz equations in (21) so that (also on account of (12)

$$p^i=\sum_{m=0}^{\infty}a_m\epsilon_m J_m(k_L^0 r)\cos m(\theta-\theta^i)\quad,\quad \text{with } \ a_m=S(\omega)e^{-im\frac{\pi}{2}}\ , \tag{23}$$

wherein $J_m(\ )$ is the $m$-th order Bessel function and $\epsilon_0=1\ ,\ \epsilon_{m>0}=2$.

The periodicity of $p^i$ entails $p^d(r,-\theta+2\theta^i,\omega)=p^d(r,\theta,\omega)$, so that on account of the first of the Helmholtz equations in (21) and the radiation condition

$$p^d=\sum_{m=0}^{\infty}b_m\epsilon_m H_m(k_L^0 r)\cos m(\theta-\theta^i)\ , \tag{24}$$

wherein $H_m(\ )=H_m^{(1)}(\ )$ is the $m$-th order Hankel function of the first kind.

The periodicity of $p^i$ and $p^d$ also entails $\phi^1(r,-\theta+2\theta^i,\omega)=\phi^1(r,\theta,\omega)$, so that on account of the second of the Helmholtz equations in (21) and the boundedness condition

$$\phi^1=\sum_{m=0}^{\infty}c_m\epsilon_m J_m(k_L^1 r)\cos m(\theta-\theta^i)\ . \tag{25}$$

By means of any one of the transmission conditions (22), and on account of the periodicity conditions satisfied by $p^i$, $p^d$, and $\phi^1$, it is found that $\psi_z^1$ obeys the relation $\psi_z^1(r,-\theta+2\theta^i,\omega)=-\psi_z^1(r,\theta,\omega)$ so that on account of the third of the Helmholtz equations in (21) and the boundedness condition

$$\psi_z^1=\sum_{m=0}^{\infty}d_m\epsilon_m J_m(k_T^1 r)\sin m(\theta-\theta^i)\ . \tag{26}$$

## 6 Use of the transmission boundary conditions to obtain $\{b_m\}$, $\{c_m\}$, $\{d_m\}$

We employ the orthogonality relations

$$\int_{\theta^i}^{\theta^i+\pi}\cos m(\theta-\theta^i)\cos n(\theta-\theta^i)\frac{d\theta}{\pi}=\frac{\delta_{mn}}{\epsilon_m}\quad,\quad\int_{\theta^i}^{\theta^i+\pi}\sin m(\theta-\theta^i)\sin n(\theta-\theta^i)\frac{d\theta}{\pi}=\delta_{mn}\frac{(1-\delta_{m0})}{2}\ , \tag{27}$$

wherein $\delta_{mm}=1\ ,\ \delta_{mn\neq m}=0$ to obtain :

$$\mathbf{P}_0\mathbf{q}_0=\mathbf{r}_0\ , \tag{28}$$

in which

$$\mathbf{P}_0=\begin{pmatrix}\chi_L^0\dot{H}_0(\chi_L^0) & -\upsilon_L^0\chi_L^1\dot{J}_0(\chi_L^1)\\ -H_0(\chi_L^0) & \upsilon_L^1 J_0(\chi_L^1)-2\mu^1(k_L^1)^2\ddot{J}_0(\chi_L^1)\end{pmatrix}\quad,\quad\mathbf{q}_0=\begin{pmatrix}b_0\\ c_0\end{pmatrix}\quad,\quad\mathbf{r}_0=\begin{pmatrix}-a_0\chi_L^0\dot{J}_0(\chi_L^0)\\ a_0J_0(\chi_L^0)\end{pmatrix}\ , \tag{29}$$

and

$$\mathbf{P}_n\mathbf{q}_n=\mathbf{r}_n\ \ ;\ \ n=1,2,...\ , \tag{30}$$

in which

$$\mathbf{P}_n = \begin{pmatrix} \chi_L^0 \dot{H}_n(\chi_L^0) & -\upsilon_L^0 \chi_L^1 \dot{J}_n(\chi_L^1) & -\upsilon_L^0 n J_n(\chi_T^1) \\ -a^2 H_n(\chi_L^0) & (\chi_L^1)^2 \left[\lambda^1 J_n(\chi_L^1) - 2\mu^1 \ddot{J}_n(\chi_L^1)\right] & 2\mu^1 n \left[J_n(\chi_T^1) - \chi_T^1 \dot{J}_n(\chi_T^1)\right] \\ 0 & 2n\left[J_n(\chi_L^1) - \chi_L^1 \dot{J}_n(\chi_L^1)\right] & -(\chi_T^1)^2 \ddot{J}_n(\chi_T^1) + \chi_T^1 \dot{J}_n(\chi_T^1) - n^2 J_n(\chi_T^1) \end{pmatrix} \tag{31}$$

$$\mathbf{q}_n = \begin{pmatrix} b_n \\ c_n \\ d_n \end{pmatrix} \quad , \quad \mathbf{r}_n = \begin{pmatrix} -a_n \chi_L^0 \dot{J}_n(\chi_L^0) \\ a_n a^2 J_n(\chi_L^0) \\ 0 \end{pmatrix} \ , \tag{32}$$

with $\chi_L^j = k_L^j a$, $\chi_T^j = k_T^j a$, $\upsilon_L^j = \lambda^j (k_L^j)^2$, $\dot{g}(\varsigma) = dg/d\varsigma$, $\ddot{g} = d^2 g/d\varsigma^2$, and for $n = 1, 2, ....$.

In principle, the matrix equations (28) and (30) enable to determine the unknown coefficient vectors $\mathbf{q}_0$ and $\mathbf{q}_n$ ; $n = 1, 2, ...$, and thus to solve the forward-scattering problem, notably for the prediction of the scattered pressure field $p_d$ in the host fluid.

Rather than do this, and since we are more interested, in the present context, in solving the *inverse-scattering problem* of the reconstruction of $\lambda^1$, $\mu^1$, $\rho^1$, we adopt a different strategy for determining $\mathbf{q}_0$ and $\mathbf{q}_n$ ; $n = 1, 2, ....$

# 7 Low-frequency approximation of the solution of the forward-scattering problem in the region outside of the body

We now define this new strategy.

We first note that :

$$\chi_L^1 = \frac{k_L^1}{k_L^0}\chi_L^0 = \frac{c_L^0}{c_L^1}\chi \quad , \quad \chi_T^1 = \frac{k_T^1}{k_L^0}\chi_L^0 = \frac{c_L^0}{c_T^1}\chi \ , \tag{33}$$

wherein $\chi := \chi_L^0$. We assume that $\chi$ is small enough (i.e., $0 < \chi << 1$) for it to be true that $0 < \chi_L^1 << 1$ and $0 < \chi_T^1 << 1$, and employ a perturbation scheme, based on the smallness of $\chi$ (which, to the very least, implies very low frequencies $\omega$ and/or small cylinder radius $a$), to solve the matrix equations.

Thus, the arguments of all the Bessel and Hankel functions appearing in the expression of $\mathbf{P}_n$ are small, which fact authorizes use to be made of the small-argument asymptotic forms

$$J_m(\xi) \sim \frac{1}{m!}\left(\frac{\xi}{2}\right)^m \ ; \ H_0^{(1)}(\xi) \sim \frac{2i}{\pi}\ln\xi \ , \ , \ H_m^{(1)}(\xi) \sim -\frac{i(m-1)!}{\pi}\left(\frac{\xi}{2}\right)^{-m} \ ; \ \xi \to 0 \ , \ m = 0, 1, ... \ , \tag{34}$$

To do this in a systematic manner, we expand $\mathbf{P}_m$, $\mathbf{q}_m$, and $\mathbf{r}_m$ in series of powers of $\chi$ :

$$\mathbf{P}_m(\chi) = \sum_{j=0}^{\infty} P_m^{(j)} \chi^j \ ; \ P_m^{(j)} := \frac{1}{j!}\frac{\partial^j}{\partial \chi^j}\mathbf{P}_m(\chi)\big|_{\chi=0} \quad , \quad \mathbf{q}_m(\chi) = \sum_{n=0}^{\infty} \mathbf{q}_m^{(n)} \chi^n \ , \tag{35}$$

$$\mathbf{r}_m(\chi) = \sum_{l=0}^{\infty} \mathbf{r}_m^{(l)} \chi^l \ ; \ \mathbf{r}_m^{(l)} := \frac{1}{l!}\frac{\partial^l}{\partial \chi^l}\mathbf{r}_m(\chi)\big|_{\chi=0} \ , \tag{36}$$

which, after introduction into the matrix equation $\mathbf{P}_m \mathbf{q}_m = \mathbf{r}_m$, yields (after comparison of powers of $\chi$) $\sum_{n=0}^{l} \mathbf{P}_m^{(l-n)} \mathbf{q}_m^{(n)} = \mathbf{r}_m^{(l)}$ ; $l = 0, 1, 2, ....$, which defines the recursive scheme for the determination of $\mathbf{q}_m^l$ :

$$\mathbf{q}_m^{(0)} = \left(\mathbf{P}_m^{(0)}\right)^{-1} \mathbf{r}_m^{(0)} \quad , \quad \mathbf{q}_m^l = \left(\mathbf{P}_m^{(0)}\right)^{-1} \left[\mathbf{r}_m^{(l)} - \sum_{n=0}^{l-1} \mathbf{P}_m^{(l-n)} \mathbf{q}_m^{(n)}\right] \; ; \; l = 1, 2, .... \; . \tag{37}$$

After a series of algebraic manipulations, the following asymptotic form of $b_m$ is found :

$$b_0 = b_0^{(2)} \chi^2 + O(\chi^4) \quad , \quad b_1 = b_1^{(2)} \chi^2 + O(\chi^4) \quad , \quad b_{m>1} = O(\chi^4) \; ; \; \chi \to 0 \; . \tag{38}$$

wherein

$$b_0^{(2)} = a_0 \left(\frac{-i\pi}{4}\right) \left[\frac{\lambda^1 + \mu^1 - \lambda^0}{\lambda^1 + \mu^1}\right] \quad , \quad b_1^{(2)} = a_1 \left(\frac{-i\pi}{4}\right) \left[\frac{\rho^0 - \rho^1}{\rho^0 + \rho^1}\right] \; , \tag{39}$$

so that the diffracted pressure field in the host fluid becomes (to second order in $\chi$)

$$p^d(r, \theta, \omega) \approx b_0^{(2)} \chi^2 H_0^{(1)}(k_L^0 r) + 2 b_1^{(2)} \chi^2 H_1^{(1)}(k_L^0 r) \cos(\theta - \theta^i) \; . \tag{40}$$

# 8 Explicit resolution of the inverse problem : recovery of the material parameters of the specimen

By making use of (40) and the first of the orthogonality relations (27) we find

$$B_m := \frac{4i}{\pi} \frac{b_m^{(2)}}{a_m} = \frac{4i}{\pi a_m \chi^2} \frac{1}{H_m^{(1)}(k_L^0 b)} \int_{\theta^i}^{\theta^i + \pi} p^d(b, \theta, \omega) \cos m(\theta - \theta^i) \frac{d\theta}{\pi} \; ; \; m = 0, 1 \; , \tag{41}$$

which signifies that $B_0$ and $B_1$ can be obtained from integrals involving the measured diffracted pressure field data (for all angles $\theta$) on a circle of radius $b$. Once these two coefficients are found, $\lambda^1$, $\mu^1$ and $\rho^1$ can, in principle, be obtained from (39), i.e.,

$$B_0 = \left[\frac{\lambda + \mu - 1}{\lambda + \mu}\right] \quad , \quad B_1 = \left[\frac{1 - \rho}{1 + \rho}\right] \; . \tag{42}$$

wherein

$$\lambda := \frac{\lambda^1}{\lambda^0} \; , \; \mu := \frac{\mu^1}{\lambda^0} \; , \; \rho := \frac{\rho^1}{\rho^0} \; . \tag{43}$$

These relations :

- show that the field is a *nonlinear* function of the material parameters, and
- apply equally-well to the fluid model of the specimen (i.e., $\mu = 0$).

Moreover, since only two pieces of data (i.e., $B_0$ and $B_1$, as expressed by (41)) are available in this (second) order of approximation (in $\chi$), only one ($\rho$), and a linear combination ($\lambda + \mu$) of the other two of the three material parameters can be recovered via (42).

More specifically, it is found that

$$\lambda + \mu = \frac{1}{1 - B_0} \; , \; \rho = \frac{1 - B_1}{1 + B_1} \; , \tag{44}$$

which underlines the fact that the proposed technique enables :

- an *explicit (partial) solution of the inverse problem* of the reconstruction of $\rho$ and $\frac{(c_L^1)^2 - (c_T^1)^2}{(c_L^0)^2} = \frac{\lambda + \mu}{\rho}$ when the material of the specimen is modeled by an elastic solid, and
- an *explicit (complete) solution of the inverse problem* of the reconstruction of $\rho$ and $\frac{(c_L^1)^2}{(c_L^0)^2} = \frac{\lambda}{\rho}$ when the material of the specimen is modeled by a fluid.

## 9 A manner for obtaining the *complete* solution of the inverse problem for the elastic solid model of the specimen

The previous analysis showed that $B_n = 0$ ; $n = 2, 3, ....$ to second order in $\chi$, which is the reason why only two pieces of data are available to reconstruct the three unknown material parameters $\rho$, $\lambda$ and $\mu$.

It can be shown that by carrying out the perturbation analysis to fourth order in $\chi$, not only $B_0$, $B_1$, but also $B_2$ are non-vanishing, so that by this means one disposes of the *three* pieces of data necessary to reconstruct the three unknown parameters.

The only difficulty with this procedure is that the relations between $B_0$, $B_1$, $B_2$ and $\rho$, $\lambda$, $\mu$ are much more complicated than previously, which fact makes it impossible to obtain explicit algebraic expressions for $\rho$, $\lambda$, $\mu$ in terms of $B_0$, $B_1$, $B_2$.

It turns out that one is faced with the problem of solving a system of three nonlinear equations in three unknowns. The procedure for solving this system is advantageously initialized via the second-order-in-$\chi$ approximations of $\lambda + \mu$ and $\rho$.

## 10 Conclusion

We have shown that the low-frequency perturbation scheme enables an *explicit reconstruction* of two ($\lambda^1 + \mu$ and $\rho$) of the three material parameters $\rho$, $\lambda$, $\mu$ of a poroelastic cylindrical specimen, modeled as an equivalent elastic solid circular cylinder.

Two single-frequency pieces of data, both involving the scattered pressure field on a complete circle around the cylinder, are required for this procedure.

Data obtained at other frequencies can be employed to reconstruct the frequency-dependent complex parameter $\lambda^1(\omega) + \mu(\omega)$ of cylindrical specimens modeled as *viscoelastic* materials.

Above all, the same type of perturbation analysis can be employed to obtain appropriate combinations of the Biot parameters when the cylindrical specimen is modeled as a fully-poroelastic material (in the sense of Biot).

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